Differential Calculi of Poincaré-Birkhoff-Witt type on Universal Enveloping Algebras

R. Martini, G.F. Post and P.H.M. Kersten
University of Twente
Department of Applied Mathematics
P.O. Box 217, 7500 AE Enschede
The Netherlands

April 12, 1995

Abstract

Differential calculi of Poincaré-Birkhoff-Witt type on universal enveloping algebras of Lie algebras g are defined. This definition turns out to be independent of the basis chosen in g. The role of automorphisms of g is explained. It is proved that no differential calculus of Poincaré-Birkhoff-Witt type exists on semi-simple Lie algebras. Examples are given, namely gl_n, Abelian Lie algebras, the Heisenberg algebra, the Witt and the Virasoro algebra. Completely treated are the 2-dimensional solvable Lie algebra, and the 3-dimensional Heisenberg algebra.

Keywords: Non-commutative differential calculus, Lie algebras, Heisenberg algebra, Virasoro algebra.

A.M.S. Subject Classification (1991): 58A10, 16S30, 81R50.

PACS: 210, 240, 0365.

1 Introduction

Recently there has been considerably interest in non-commutative differential geometry, both mathematically and as a framework for certain physical theories. In particular there is much activity in differential geometry related to quantum groups.

For instance Woronowicz [1] developed a general theory of fields and forms on quantum groups. This general theory has been reformulated by Wess and Zumino [2] in an easier accessible way. Furthermore their approach may turn out to be more suitable for explicit computations with regard to applications in physics. A number of very interesting papers (e.g. [3]-[6]) elucidating various aspects and studying specific examples have been written since.

In this paper we deal with possible differential calculi on Lie algebras. To be more precise, differential calculi on the universal enveloping algebra of a given Lie algebra are defined and we investigate the possibility to construct such a calculus. Our approach, starting from generators and relations, is somewhat related to the method of Wess and Zumino, although it is different from the

traditional approaches to construct differential calculi on quantum groups. For a Hopf algebraic interpretation of the method and results, see [7].

It can be proved that there exists no differential calculus in this sense on any semisimple Lie algebra. However for non-semisimple Lie algebras things turn out differently. For example there exists a number of interesting non-trivial differential calculi on any general linear algebra, the Heisenberg algebra and also on the infinite-dimensional Virasoro algebra.

2 Conditions for differential calculi of Poincaré-Birkhoff-Witt type on the universal enveloping algebra of a Lie algebra

In this section our aim is to determine conditions for differential calculi of Poincaré-Birkhoff-Witt type on the universal enveloping algebra of a given Lie algebra g. Let us start to describe what we mean by this. If x^i, $(i \in I)$ is a basis of g, and $\preccurlyeq$ is a well-ordering on I, then the Poincaré-Birkhoff-Witt theorem tells that

$$x^{i_1} x^{i_2} \dots x^{i_m} \qquad i_1 \preccurlyeq i_2 \preccurlyeq \dots \preccurlyeq i_m$$

is a basis of the universal enveloping algebra $\mathcal{U}_g$. We shall use the following

Definition 2.1 *By a differential calculus of Poincaré-Birkhoff-Witt type we mean an algebra extension $\mathcal{E}$ of $\mathcal{U}_g$, such that*

(a) $\mathcal{E} = \oplus_{n=0}^{\infty} \mathcal{E}^n$ *is a grading of* $\mathcal{E}$, *with* $\mathcal{E}^0 = \mathcal{U}_g$.

(b) There is a linear map d: $\mathcal{E} \to \mathcal{E}$, $d(\mathcal{E}^n) \subset \mathcal{E}^{n+1}$ *that satisfies*

(1) $d^2 = 0$.

(2) $d(\omega\eta) = d(\omega)\eta + (-1)^n \omega d\eta$ *for* $\omega \in \mathcal{E}^n$.

(c) If $y^i = d(x^i)$ *then the elements*

$$y^{j_1} y^{j_2} \dots y^{j_n} x^{i_1} x^{i_2} \dots x^{i_m}$$

with $j_1 \prec \dots \prec j_n$ *and* $i_1 \preccurlyeq i_2 \preccurlyeq \dots \preccurlyeq i_m$ *form a (linear) basis of* $\mathcal{E}^n$.

Due to these requirements we have the following relations in $\mathcal{E}$. First we have

$$x^i x^j = x^j x^i + c_k^{ij} x^k \qquad (j \prec i) \tag{2.1}$$

where c_k^{ij} are the structure constants of the given Lie algebra g. Throughout the summation convention is applied.

Secondly we have

$$x^i y^j = y^j x^i + y^k A_k^{ij} \qquad (i, j \in I) \tag{2.2}$$

We assume that A_k^{ij} are (complex) numbers. In the case that all $A_k^{ij} = 0$, the relations (2.2) represent the commutation relations of the standard calculus. Our task will be to derive conditions for the A_k^{ij} (expressed in terms of c_k^{ij}) in order to get a calculus of Poincaré-Birkhoff-Witt type. One might object that the requirement that the A_k^{ij} are constant is rather severe. However, one can prove that adding non-zero terms, linear in x^i makes it impossible to define a Hopf algebra structure on $\mathcal{E}$, see [7]. This aspect we will discuss here any further.

Finally we have, by applying d to the relations (2.2), and using the requirements from definition 2.1 (b)

$$y^i y^j = -y^j y^i \quad (j \prec i) \qquad \text{and} \qquad y^i y^i = 0 \tag{2.3}$$

We introduce an ordering on the monomials $z^{i_1} z^{i_2} \ldots z^{i_m}$, where z is either x or y. First we order by length, and after that lexicographically, with x^i larger then y^j. Doing this the relations (2.1), (2.2) and (2.3) can be seen as rewrite-rules, replacing the leading term (in the left-hand side) by lower terms. With respect to these rewrite-rules we have the "normal monomials", i.e. the monomials that can not be simplified using the rewrite-rules (2.1), (2.2) and (2.3). It is easy to see that the elements in definition 2.1 (c) are exactly the normal monomials. Now we can apply the diamond lemma (see i.e. [8]). It states the following: if rewriting $(z^i z^j) z^k$ and $z^i(z^j z^k)$ lead to the same normal form, then the normal monomials are a basis of $\mathcal{E}$. Here $z = x$ or $z = y$ with $z^k \prec z^j \prec z^i$. So we have the following steps:

I. Consider $(x^i x^j) x^k$ and $x^i(x^j x^k)$. In this case Jacobi's identity guarantees the same normal forms. (It is in fact a way to prove the Poincaré-Birkhoff-Witt theorem).

II. Consider $(x^i x^j) y^k$ and $x^i(x^j y^k)$. We have

$$\begin{aligned} x^i(x^j y^k) &= x^i(y^k x^j + y^s A^{jk}_s) \\ &= (y^k x^i + y^r A^{ik}_r) x^j + (y^s x^i + y^r A^{is}_r) A^{jk}_s \\ &= y^k x^i x^j + y^r (A^{ik}_r x^j + A^{jk}_r x^i + A^{is}_r A^{jk}_s) \\ &= y^k (x^j x^i + c^{ij}_s x^s) + y^r (A^{ik}_r x^j + A^{jk}_r x^i + A^{is}_r A^{jk}_s) \end{aligned}$$

and

$$\begin{aligned} (x^i x^j) y^k &= (x^j x^i + c^{ij}_s x^s) y^k \\ &= y^k x^j x^i + y^r (A^{jk}_r x^i + A^{ik}_r x^j + A^{js}_r A^{ik}_s) + c^{ij}_s (y^k x^s + y^r A^{sk}_r) \end{aligned}$$

from which it follows that we need

$$A^{jk}_s A^{is}_r - A^{ik}_s A^{js}_r = c^{ij}_s A^{sk}_r \tag{2.4}$$

III. Consider $(x^i y^j) y^k$ and $x^i(y^j y^k)$. We have

$$\begin{aligned} (x^i y^j) y^k &= (y^j x^i + y^s A^{ij}_s) y^k \\ &= y^j (y^k x^i + y^r A^{ik}_r) - A^{ij}_s y^k y^s \\ &= -y^k y^j x^i + A^{ik}_r y^j y^r - A^{ij}_s y^k y^s \end{aligned}$$

and

$$\begin{aligned} x^i(y^j y^k) &= -x^i(y^k y^j) \\ &= -(y^k x^i + y^r A^{ik}_r) y^j \\ &= -y^k (y^j x^i + y^s A^{ij}_s) + A^{ik}_r y^j y^r \end{aligned}$$

From this it is clear that $(x^i y^j) y^k$ reduces to the same normal form as $x^i(y^j y^k)$.

IV. Consider $(y^i y^j) y^k$ and $y^i(y^j y^k)$. Both reduce to $-y^k y^j y^i$, so no additional conditions arise.

To summarize we derived that the monomials in definition 2.1 form indeed a basis of $\mathcal{E}$ if (and only if) the equations (2.4) are satisfied.

Till now we didn't take care of d. Let us describe how this can be done. Consider the tensor algebra on the letters x^i and y^j, $i, j \in I$. A basis are the monomials

$$z^{i_1} z^{i_2} \ldots z^{i_m}$$

where $i_1, i_2, \ldots, i_m \in I$ (not necessarily ordered) and $z = x$ or $z = y$. On the tensor algebra it is easy to define d, namely

$$\mathrm{d}(z^{i_1} z^{i_2} \ldots z^{i_m}) = \sum_{s=1}^{m} (-1)^{d_s - 1} z^{i_1} z^{i_2} \ldots \mathrm{d}(z^{i_s}) \ldots z^{i_m}$$

Here d_s is the number of times that $z = y$ in $z^{i_1} z^{i_2} \ldots z^{i_{s-1}}$. Of course $\mathrm{d}(x^i) = y^i$ and $\mathrm{d}(y^j) = 0$. It is not difficult to prove that d satisfies the requirements in definition 2.1 (b). Now we want to define d on $\mathcal{E}$. For this it is sufficient to prove that d preserves the relations of (2.1), (2.2) and (2.3), since in that case d leaves the ideal generated by (2.1), (2.2) and (2.3) invariant, thanks to $\mathrm{d}(\omega\eta) = \mathrm{d}(\omega)\eta + (-1)^n \omega \mathrm{d}\eta$ in the tensor algebra. Now applying d to the relations (2.2) yields the relations (2.3) (like before), and applying d to the relations (2.3) yields 0. Hence only the relations (2.1) remain. We have

$$\begin{aligned} \mathrm{d}(x^i x^j - x^j x^i) &= y^i x^j + x^i y^j - y^j x^i - x^j y^i \\ &= y^i x^j + (y^j x^i + A_k^{ij} y^k) - y^j x^i - (y^i x^j + A_k^{ji} y^k) \\ &= (A_k^{ij} - A_k^{ji}) y^k = c_k^{ij} y^k \end{aligned}$$

So we find

$$A_k^{ij} - A_k^{ji} = c_k^{ij} \tag{2.5}$$

In the next section we will study these equations (2.4) and (2.5) more closely. In particular, we will show that they may be interpreted Lie algebraically in a very interesting way.

3 Lie algebraic interpretation of the equations for a differential calculus

The conditions to be satisfied in order to obtain a consistent differential calculus are the equations (2.4) and (2.5).

Denote by g the Lie algebra with basis $x^i (i \in I)$ and with structure constants c_k^{ij}. Now define a linear mapping $\rho : g \to gl(g)$ by

$$\rho(x^i) x^a = A_b^{ia} x^b$$

From (2.4) it follows that

$$(A_b^{ia} A_c^{jb} - A_b^{ja} A_c^{ib}) x^c = -c_k^{ij} A_c^{ka} x^c$$

and thus

$$A_b^{ia} \rho(x^j) x^b - A_b^{ja} \rho(x^i) x^b = -c_k^{ij} \rho(x^k) x^a$$

Therefore

$$\rho(x^j) A_b^{ia} x^b - \rho(x^i) A_b^{ja} x^b = -\rho(c_k^{ij} x^k) x^a$$

yielding

$$\rho(x^j)\rho(x^i) x^a - \rho(x^i)\rho(x^j) x^a = \rho(c_k^{ji} x^k) x^a = \rho([x^j, x^i]) x^a$$

So

$$\rho(x^j)\rho(x^i) - \rho(x^i)\rho(x^j) = \rho([x^j, x^i])$$

and therefore in general we have

$$\rho(x)\rho(y) - \rho(y)\rho(x) = \rho([x, y]) \tag{3.1}$$

which means that the linear mapping ρ is a representation of the Lie algebra g on itself.

In fact it is clear that ρ should be a representation from the beginning. Namely consider the linear span $Y = \langle y^i; i \in I\rangle$. Then ρ can be considered to be a mapping $\rho : g \to gl(Y)$, given by

$$\rho(x)y = [x, y] \qquad x \in g, y \in Y$$

as $x^i y^j - y^j x^i = A^{ij}_k y^k$. Clearly we have

$$[[x, \tilde{x}], y] = [x, [\tilde{x}, y]] - [\tilde{x}, [x, y]]$$

and consequently

$$\rho([x, \tilde{x}]) = \rho(x)\rho(\tilde{x}) - \rho(\tilde{x})\rho(x).$$

Next we consider the other condition. From the equation (2.5) it follows that

$$A^{ij}_k x^k - A^{ji}_k x^k = c^{ij}_k x^k$$

yielding

$$\rho(x^i)x^j - \rho(x^j)x^i = [x^i, x^j]$$

and therefore in general

$$\rho(x)y - \rho(y)x = [x, y] \tag{3.2}$$

Thus we see that in order to determine a consistent differential calculus we should look for representations ρ of the given Lie algebra on itself which moreover satisfy the condition (3.2).

This interpretation allows us to describe all constructions till now in a coordinate-independent way. Let us start with a linear space g. Then we found that the differential calculi above are completely described by two mappings, $c : g \times g \to g$ and $\rho : g \to gl(g)$ such that for all $x, y, z \in g$:

1. $c(x, y) = -c(y, x)$ and $c(c(x, y), z) + c(c(y, z), x) + c(c(z, x), y) = 0$
   i.e. g is a Lie algebra,

2. $\rho(c(x, y)) = \rho(x)\rho(y) - \rho(y)\rho(x)$ and $\rho(x)y - \rho(y)x = c(x, y)$

After choosing coordinates (x^i) we have structure constants c^{ij}_k instead of c and the numbers A^{ij}_k instead of ρ. In a different basis $(\bar{x}^i)$ the structure constants will be different, say $\bar{c}^{ij}_k$, and similarly $\bar{A}^{ij}_k$. If τ is the change of coordinates, i.e. $\bar{x}^i = \tau^i_k x^k$ and σ the inverse, then we have

$$\bar{c}^{ij}_k = \tau^i_s \tau^j_r \sigma^t_k c^{sr}_t \quad \text{and} \quad \bar{A}^{ij}_k = \tau^i_s \tau^j_r \sigma^t_k A^{sr}_t \tag{3.3}$$

A special case arises if τ is an automorphism of the Lie algebra g. Then $\bar{c}^{ij}_k = c^{ij}_k$, while not necessarily $\bar{A}^{ij}_k = A^{ij}_k$. So if (c^{ij}_k, A^{ij}_k) is a solution, so is $(c^{ij}_k, \bar{A}^{ij}_k)$, with $\bar{A}^{ij}_k$ given by (3.3) and τ an automorphism. We will call those solutions equivalent and will content ourselves determining one solution from each equivalence class.

The transition from A^{ij}_k to $\bar{A}^{ij}_k$ can be described on the level of ρ quite easily. One can calculate directly that

$$\bar{\rho}(x) = \tau^{-1}\rho(\tau x)\tau \tag{3.4}$$

Hence we will be interested in all solutions ρ up to an automorphism in the sense of equation (3.4).

4 Semisimple Lie algebras and gl_n

In this section we will prove that there exists no differential calculus of the form described in section 2 on the universal enveloping algebra of any semisimple Lie algebra g.

To this end we have to prove that there exists no representation ρ of g on itself, which moreover satisfies

$$[x, y] = \rho(x)y - \rho(y)x \qquad \text{for all } x, y \in g. \tag{4.1}$$

Let ρ be any representation of g on itself. One can interpret equation (4.1) in cohomological terms by saying that the identity $i : g \to g$ is a cocycle in the complex $C^*(g; \rho)$. According to the first Whitehead lemma, we know that $H^1(g; \rho) = 0$ for any semisimple g and representation ρ. Therefore i is a coboundary, i.e. there exists an element $a \in C^0(g; \rho) = g$ such that $\mathrm{d}(a) = i$. Hence it follows that

$$i(x) = \rho(x)a, \qquad \text{or } x = \rho(x)a \tag{4.2}$$

Now set $y = a$ in equation (4.1) and substitute $\rho(x)a = x$. We arrive at $[x, a] = x - \rho(a)x$ or equivalently

$$\rho(a) = i + \mathrm{ad}\ a \tag{4.3}$$

Here ad denotes the adjoint representation. Taking traces of the mappings above, we see that $\mathrm{tr}(\rho(a)) = \dim(g)$, as $\mathrm{tr}(\mathrm{ad}\ a) = 0$. Hence a is represented by a matrix $\rho(a)$ which is not traceless. This is impossible for semisimple Lie algebras.

For gl_n things turn out differently. Indeed in this case one can find several solutions, of which we describe one. One can take

$$\rho(x)y = xy$$

where in the right-hand side xy denotes the multiplication of x and y as $n \times n$-matrices. It is obvious that this is a representation satisfying $[x, y] = \rho(x)y - \rho(y)x$. Details for $n = 2$ may be found in [12].

5 Abelian Lie algebras

In this section we suppose that the Lie algebra g is Abelian of dimension n. The equations read

$$[\rho(x^i), \rho(x^j)] = 0 \qquad \text{and} \qquad \rho(x^i)x^j = \rho(x^j)x^i \tag{5.1}$$

Hence the matrices $\rho(x^i)$ form a system of commuting matrices satisfying an additional condition. In low dimensional cases ($\dim g \leq 6$) maximal commutative matrix algebras have been determined explicitly (see e.g. [9]).

Thus for small dimensions all differential calculi of the form (2.2) on universal enveloping algebras of Abelian Lie algebras, actually affine spaces, can be described explicitly. For dimensions $n = 1, 2$ this has already been done by Dimakis et al [10]. See also [11].

In general we can describe the "regular" case. By this we mean that all matrices are simultaneous diagonalizable and linearly independent. In this case $\rho(x^i) = E^i_i$ with respect to the basis $\{x^1, \ldots, x^n\}$. Here E^i_j denotes the matrix with the (j,i)-entry equal to 1, and all others 0. The corresponding calculus is defined by

$$x^i y^j = y^j x^i + \delta^j_i y^j.$$

6 The 2-dimensional solvable Lie algebra

Here we are concerned with the smallest non-Abelian (solvable) Lie algebra g, which is up to an isomorphism given by the structure constants $c_2^{12} = 2$. We will denote $x^1 = x$ and $x^2 = y$, and we have $[x, y] = 2y$. Our task is to find representations ρ of g on g itself, satisfying $\rho(x)y - \rho(y)x = 2y$. We will perform these computations in some detail. For the time being we forget the second condition; we calculate $\rho(x)$ and $\rho(y)$ with respect to a suitable basis. These "standard forms" can be improved by applying automorphisms of g. We are just interested to determine one element in each orbit $\bar{\rho}(g) = \tau^{-1}\rho(\tau g)\tau$, as we explained in section 3. We see that instead of $\rho(g)$ we can take $\rho(\tau g)$ as long as we are only interested in a standard form for $\rho(g)$ with respect to some basis; indeed the conjugation with τ can be seen as merely a basis-transformation.

With respect to some basis, we can assume that $\rho(y)$ has Jordan normal form. From $[x, y] = 2y$ we see that $\operatorname{tr}(\rho(y)) = 0$. Hence we have two cases, case A and case B.

Case A.

$$\rho(y) = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}$$

From $[\rho(x), \rho(y)] = 2\rho(y)$ it follows that $\alpha = 0$, so that $\rho(y) = 0$. So now we can put $\rho(x)$ in Jordan normal form. We have two subcases, A1 and A2.

Case A1.

$$\rho(y) = 0 \qquad \text{and} \qquad \rho(x) = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

Now the second condition comes in. We put

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \text{and} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

then $\rho(x)y - \rho(y)x = 2y$ yields $y_1(\alpha - 2) = 0$ and $y_2(\beta - 2) = 0$. Working this out (splitting with respect to y_1 and y_2 vanishing or not) yields one solution

$$\rho(y) = 0 \qquad \text{and} \qquad \rho(x) = \begin{pmatrix} \alpha & 0 \\ \beta(\alpha - 2) & 2 \end{pmatrix} \tag{6.1}$$

Here and hereafter the matrices with round brackets (and) are with respect to the basis $\{x, y\}$. Finally we can apply the automorphism τ, given by

$$\tau = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}$$

so that $\bar{\rho}(x) = \tau^{-1}\rho(\tau x)\tau$ takes the same form as in (6.1) but with $\beta = 0$.

Case A2.

$$\rho(y) = 0 \qquad \text{and} \qquad \rho(x) = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$$

Now $\rho(x)y - \rho(y)x = 2y$ yields $y_2 = -y_1(\alpha - 2)$ and $y_2(\alpha - 2) = 0$. It follows that $y_1 \neq 0$ since otherwise $y = 0$. So $y_2(\alpha - 2) = -y_1(\alpha - 2)^2 = 0$ yields $\alpha = 2$. Hence $y_2 = 0$. This leads to a solution of the form

$$\rho(y) = 0 \qquad \text{and} \qquad \rho(x) = \begin{pmatrix} 2 & 0 \\ \beta & 2 \end{pmatrix} \text{ with } \beta = \frac{x_2}{y_1} \neq 0 \tag{6.2}$$

Applying $\tau = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}$ yields the same form as in (6.2) but now with $\beta = 1$.

Case B.

$$\rho(y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and therefore } \rho(x) = \begin{bmatrix} \alpha+1 & \beta \\ 0 & \alpha-1 \end{bmatrix}$$

Applying $\tau = \begin{pmatrix} 1 & 0 \\ -\beta & 1 \end{pmatrix}$ we see that we can take $\beta = 0$. The condition $\rho(x)y - \rho(y)x = 2y$ yields is this case $x_2 = (\alpha-1)y_1$ and $y_2(\alpha-3) = 0$. Hence case B splits in two cases

Case B1. $y_2 = 0$ and hence $y_1 \neq 0$ and $\alpha \neq 1$.
We obtain

$$\rho(y) = \begin{pmatrix} 0 & 0 \\ \alpha-1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(x) = \begin{pmatrix} \alpha-1 & 0 \\ \beta & \alpha+1 \end{pmatrix} \tag{6.3}$$

with $\beta = 2\frac{x_1}{y_1}$. For $\alpha \neq -1$ we can apply $\tau = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$ with $\gamma = -\frac{\beta}{\alpha+1}$ to get $\rho(x)$ in the same form as in (6.3) with $\beta = 0$. However for $\alpha = -1$ we have a bifurcation. Either $\beta = 0$ and we have the same form as above, or $\beta \neq 0$. In this case one can scale such that $\beta = 1$. So we have an extra solution here:

$$\rho(y) = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(x) = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.4}$$

Finally we come to the last case.

Case B2. $y_2 \neq 0$ and hence $\alpha = 3$.
This case yields a 2-parameter family of solutions, which by a automorphism can be put in the form

$$\rho(y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(x) = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \tag{6.5}$$

Concluding we have 5 families of solutions up to automorphisms; *i*: (6.1) with $\beta = 0$, *ii*: (6.2) with $\beta = 1$, *iii*: (6.3) with $\beta = 0$, *iv*: (6.4) and *v*: (6.5).

To prove that these classes are indeed not equivalent, one can use fruitfully the invariance of $\mathrm{rank}(\rho(y))$ and $\mathrm{tr}(\rho(x))$ under automorphisms.

7 The Heisenberg algebra

In this section we are concerned with the Heisenberg algebra generated by $2n+1$ generators p_i, q_i $(i = 1, \ldots, n)$ and c subjected to the relations

$$[p_i, q_i] = c \quad (i = 1, \ldots, n) \tag{7.1}$$

And all other commutators are 0.

We will describe the calculi for $n = 1$ in detail. We will write $x^1 = c, x^2 = p_1$ and $x^3 = q_1$ in this case. Thus $c_1^{23} = 1, c_1^{32} = -1$ and all other structure constants vanish. The calculation of all solutions ρ up to automorphisms can be performed in the same way as in section 6. The result can be divided into 4 groups of which it is clear that they are mutually not equivalent under automorphisms; we give ρ with respect to the basis $\{c, p, q\}$.

I. $\rho(c) \neq 0$. Then (with $\epsilon = 0$ or $\epsilon = 1$)

$$\rho(c) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \rho(p) = \begin{pmatrix} 0 & \epsilon & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(q) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

II. $\rho(c) = 0$ and ρ is not traceless. Then (with $\epsilon = 0$ or $\epsilon = 1$)

$$\rho(p) = \begin{pmatrix} 0 & \epsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(q) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

III. $\rho(c) = 0$ and ρ is traceless. Then (with $\alpha \in \mathbb{C}$) either

$$\rho(p) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(q) = \begin{pmatrix} 0 & -1 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

or

$$\rho(p) = \begin{pmatrix} 0 & 0 & 1+\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(q) = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

IV. $\rho(c) = 0$ and the solution is invariant under automorphisms.

$$\rho(p) = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(q) = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remarkably the solution in IV is the only solution invariant under automorphisms. It corresponds to taking $A^{ij}_k = \frac{1}{2}c^{ij}_k$. Indeed it is a solution, which can be generalized to higher dimensional Heisenberg algebras by the same relation to c^{ij}_k; from equation (3.3) it is clear that also in higher dimensions it is preserved by automorphisms.

The corresponding consistent differential calculus satisfies all commutation relations of classical calculus with the exception that

$$\begin{aligned} p_i \,\mathrm{d}q_i &= \mathrm{d}q_i \, p_i + \tfrac{1}{2}\mathrm{d}c \\ q_i \,\mathrm{d}p_i &= \mathrm{d}p_i \, q_i - \tfrac{1}{2}\mathrm{d}c \end{aligned} \tag{7.2}$$

Likewise we have for the basic differential operators $\partial_c, \partial_{p_i}, \partial_{q_i}$, besides the classical, the non-traditional commutation relations with the generators

$$\begin{aligned} \partial_c p_i &= p_i \partial_c + \tfrac{1}{2}\partial_{q_i} \\ \partial_c q_i &= q_i \partial_c - \tfrac{1}{2}\partial_{p_i} \end{aligned} \tag{7.3}$$

8 The Witt algebra and the Virasoro algebra

We consider the algebra W generated by generators $x^i (i \in \mathbb{Z})$ subjected to the relations

$$x^i x^j - x^j x^i = (j-i) x^{i+j} \tag{8.1}$$

W is called the Witt algebra. We set

$$A_k^{ij} = (j+\mu)\delta_k^{i+j}, \tag{8.2}$$

where μ is a complex number. With this choice A_k^{ij} satisfies (2.4) and (2.5). Indeed,

$$A_k^{ij} - A_k^{ji} = (j+\mu)\delta_k^{i+j} - (i+\mu)\delta_k^{i+j} = (j-i)\delta_k^{i+j} = c_k^{ij}$$

and

$$\begin{aligned} A_r^{ik} A_l^{jr} - A_r^{jk} A_l^{ir} &= (k+\mu)\delta_r^{i+k}(r+\mu)\delta_l^{j+r} - (k+\mu)\delta_r^{j+k}(r+\mu)\delta_l^{i+r} \\ &= (k+\mu)(i+k+\mu)\delta_l^{i+j+k} - (k+\mu)(j+k+\mu)\delta_l^{i+j+k} \\ &= (k+\mu)(i-j)\delta_l^{i+j+k} = (i-j)A_l^{i+j,k} = -c_s^{ij} A_l^{sk} \end{aligned}$$

The corresponding calculus satisfies

$$x^i \mathrm{d}x^j = \mathrm{d}x^j x^i + (j+\mu)\mathrm{d}x^{i+j} \tag{8.3}$$

and for the basic differential operators $\partial_p (p \in \mathbb{Z})$ we have the commutation relations with the generators

$$\partial_p x^k = \delta_p^k + x^k \partial_p + (p-k+\mu)\partial_{p-k} \tag{8.4}$$

The Virasoro algebra V is obtained from the algebra W by central extension with a central element t. Then the relations become

$$x^n x^m - x^m x^n = (m-n)x^{n+m} + \frac{1}{12}(m^3 - m)\delta^{m,-n} t \tag{8.5}$$

In case of the Virasoro algebra we put

$$A_k^{n,m} = (m+\mu)\delta_k^{n+m} \quad \text{if } n, m, k \in \mathbb{Z} \tag{8.6}$$

and

$$A_l^{t,j} = A_l^{i,t} = A_t^{i,j} = 0$$

with the exception that

$$A_t^{m,-m} = \frac{1}{24}(m^3 - m).$$

This modification of A_k^{ij} satisfies again the relations (2.4) and (2.5) and we obtain a calculus with the property that for $n, m \in \mathbb{Z}$ we have

$$x^n \mathrm{d}x^m = \mathrm{d}x^m x^n + (m+\mu)\mathrm{d}x^{n+m} + \frac{1}{24}(m^3 - m)\delta^{n,-m}\mathrm{d}t \tag{8.7}$$

and for $k, p \in \mathbb{Z}$

$$\begin{aligned} \partial_p x^k &= \delta_p^k + x^k \partial_p + (p-k+\mu)\partial_{p-k} \\ \partial_p t &= t\partial_p \\ \partial_t x^k &= x^k \partial_t + \tfrac{1}{24}(k^3 - k)\partial_{-k} \\ \partial_t t &= 1 + t\partial_t \end{aligned}$$

We hope to report on applications to differential equations in a future publication.

References

[1] Woronowicz, S.L.
Differential calculus on compact matrix pseudogroups (quantum groups), Commun. Math. Phys. **122**, 125 (1989).

[2] Wess, J. and Zumino, B.
Covariant differential calculus on the quantum hyperplane, Nucl. Phys. **188**, 303-312 (1990).

[3] Schupp, P., Watts, P. and Zumino, B.
Differential geometry on linear quantum groups, Lett. Math. Phys. **25**, 139-147 (1992).

[4] Jurco, B.
Differential calculus on quantized simple Lie groups, Lett. Math. Phys. **22**, 177-186 (1991).

[5] Aschieri, P. and Castellani, L.
An introduction to non-commutative differential geometry on quantum groups, Preprint CERN-TH. 6565/92.

[6] Maltsiniotis, G.
Groupes quantiques et structures différentielles, C.R. Acad. Sci., t. 311, Série I, 831-834 (1990).

[7] Van den Hijligenberg, N.W. and Martini, R.
A differential Hopf algebra on the universal envelope of a Lie algebra, preprint CWI, Amsterdam (1995).

[8] Bergman, G.M.
The diamond lemma for ring theory, Adv. Math. **29**, 178-218 (1978).

[9] Suprunenko, D.A. and Tyshkevich, R.I.
Commutative Matrices, Academic Press, New York, (1968).

[10] Dimakis, A., Müller-Hoissen, F. and Striker, T.
Non-commutative differential calculus and lattice gauge theory, J. Phys. A: Math. Gen. **26**, 1927-1949 (1993).

[11] Baehr, H.C., Dimakis, A. and Müller-Hoissen, F.
Differential calculi on commutative algebras, hep-th/9412069 (1994).

[12] Martini, R. and Post, G.
Differential Calculi on $\mathfrak{U}(sl_2)$, Preprint University of Twente (1995).